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Plane deformations of shear-resisting membranes formed of elastic cords

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Abstract. The governing equations are formulated and some exact solutions are obtained for plane deformation of membranes formed of two families of elastic cords. The cords are assumed to be continuously distributed and every cord of one family is joined to each cord of the other family at their point of intersection. The membranes are incapable of withstanding in-plane compression but they exhibit shear resistance and a general nonlinear stretch-tension relation. The solutions include deformations with constant tensions (or stretches), deformations with straight cords and a half-universal state of tensions which satisfies the governing equations for any stretch-tension relations but a particular shear-deformation relation.

1. Introduction

The continuum theory for plane deformations of networks formed by two families of continuously distributed inextensible cords was first formulated by Rivlin [1]. The cords are continuously distributed so that the networks can be treated as membranes. The cords of different families are joined together at their points of intersection so that there is no slip between them. It was also assumed that the cords can withstand tension but cannot transmit compression. Rivlin's theory, which assumes no shearing resistance between the cords, was subsequently applied by Rogers and Pipkin [2] to treat problems of inextensible networks with holes. An extension of Rivlin's model to include shear effects was proposed by Pipkin [3, 4], who also discussed some of singularities that may occur in the solutions. A later paper by Pipkin [5] deals with a modification of Rivlin's theory so that the cords may shorten but not lengthen and may transmit tension but not compression.

The continuum theory for networks formed by two families of straight elastic cords has been formulated by Genensky and Rivlin [6]. They have obtained solutions to the displacement boundary value problem, the traction boundary value problem and the mixed boundary value problem under the restriction of linear elastic response and infinitesimal strains.

When finite strains in the elastic cords are allowed, the author [7] has found some analytic solutions to plane deformation in both discrete networks and continuous membranes. He also discussed some degenerate deformations of continuous membranes such as slack regions, in which the cords of one and/or both families are unstretched, and the uniqueness of the deformation in one class of membranes. Consequently three papers have resulted (see Green and Shi [8, 9, 10]). In addition, he obtained some numerical solutions and some extra analytic solutions for the continuous membranes. These involve out-of-plane deformations of the membranes with straight cords, plane deformations of the membranes with curvilinear cords and plane deformations of the membranes with straight cords and shear resistance. It is the last part that forms the major part of this paper. The membrane formed in this way could be used to model cloth.

In Section 2 we set up the equations governing the plane deformations of the membranes with shear resistance. In Section 3 we first discuss the deformation in which a finite region of the membrane collapses into a single curve. It is found that the collapse curve must be a straight line if it transmits load. Then we show that the homogeneous deformation is the only deformation in which the tensions (or the stretches) are constant. Section 4 is devoted to inverse solutions in which the cords in one family remain straight and parallel while those in the other are deformed into a family of congruent curves. The state of tensions in these solutions is shown to be half-universal since the equilibrium and compatibility equations are solved explicitly by specifying the shear-deformation relation only without stretch-tension relations. In Section 5 we consider a particular membrane with linear relation between tension and stretch. The deformations investigated are those in which one family of cords remains straight and parallel while the other just remains straight. Finally, in Section 6 we discuss some special deformations in slack regions in which one of two or both tensions vanishes. These solutions are valid for general relations between the stress and deformation.

2. Governing equations

We consider a plane membrane formed of two families of straight parallel elastic cords with the cords of one family being initially orthogonal to the cords of the other. When we refer to a membrane here we have assumed that the cords are distributed so closely that it can be treated as a continuum. The membrane is not capable of withstanding in-plane compression but able to resist shear. Let Ox_1x_2 be a system of plane Cartesian coordinates with axes parallel to the initial directions of the cords and consider a plane deformation in which a material point with initial coordinates (X_1, X_2) relative to this system has coordinates (x_1, x_2) in the deformed configuration, where

$$x_1 = x_1(X_1, X_2)\,, \qquad x_2 = x_2(X_1, X_2)\,. \tag{2.1}$$

We refer to the family of cords $X_2 =$ constant as the X_1-cords and the family $X_1 =$ constant as the X_2-cords. The deformation gradient tensor and Green–Lagrange strain tensor (see Spencer [11] and Ogden [12]) are

$$\mathbf{F} = \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}, \tag{2.2}$$

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} x_{1,1}^2 + x_{2,1}^2 & x_{1,1}x_{1,2} + x_{2,1}x_{2,2} \\ x_{1,1}x_{1,2} + x_{2,1}x_{2,2} & x_{1,2}^2 + x_{2,2}^2 \end{pmatrix}, \tag{2.3}$$

where $x_{\alpha,\beta} = \partial x_\alpha / \partial X_\beta$ $(\alpha,\ \beta = 1, 2)$.

According to the theory of continuum mechanics, a line element along the X_1-cord, or an element of the X_1-cord, at the point (X_1, X_2) is stretched to $\lambda_1 = (C_{11})^{1/2} = (x_{1,1}^2 + x_{2,1}^2)^{1/2}$ times its original length and rotated to the direction of the unit vector $\mathbf{a}$ given by

$$\mathbf{a} = (x_{1,1}\mathbf{i} + x_{2,1}\mathbf{j})/\lambda_1\,,$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along Ox_1 and Ox_2, respectively (see Spencer [11]).

Letting φ denote the angle between **a** and the unit vector **i** we then have that

$$x_{1,1} = \lambda_1 \cos \varphi , \qquad x_{2,1} = \lambda_1 \sin \varphi . \tag{2.4}$$

Similarly for a line element in the direction of the X_2-cord at the point (X_1, X_2), we have the stretch $\lambda_2 = (C_{22})^{1/2} = (x_{1,2}^2 + x_{2,2}^2)^{1/2}$, the deformed direction along the unit vector **b** given by

$$\mathbf{b} = (x_{1,2}\mathbf{i} + x_{2,2}\mathbf{j}) / \lambda_2 ,$$

and

$$x_{1,2} = \lambda_2 \cos \psi , \qquad x_{2,2} = \lambda_2 \sin \psi , \tag{2.5}$$

where ψ is the angle between **b** and the x_1-axis.

From (2.4) and (2.5) we get the compatibility conditions for the stretches λ_1, λ_2 and the directions represented by φ and ψ

$$\begin{cases} \dfrac{\partial}{\partial X_2} (\lambda_1 \cos \varphi) = \dfrac{\partial}{\partial X_1} (\lambda_2 \cos \psi) , \\ \dfrac{\partial}{\partial X_2} (\lambda_1 \sin \varphi) = \dfrac{\partial}{\partial X_1} (\lambda_2 \sin \psi) . \end{cases} \tag{2.6}$$

Here we should note that both λ_1 and λ_2 are not, in general, the principal stretches of the deformation, since they are not the eigenvalues of the Green–Lagrange strain tensor. To avoid the membrane turning over, we assume that $0 \leqslant \psi - \varphi \leqslant \pi$.

Since the membrane is able to resist shear, the force carried by a cord is not necessarily in the direction of the cord. We denote by $\mathbf{T}_a$ the force carried by the X_1-cords crossing unit initial length of the X_2-cord, with $\mathbf{T}_b$ similarly defined for the X_2-cords at the point (X_1, X_2). If we project the forces in the directions **a** and **b**, then we have

$$\begin{cases} \mathbf{T}_a = T_1\mathbf{a} + S\mathbf{b} = (T_1 \cos \varphi + S \cos \psi)\mathbf{i} + (T_1 \sin \varphi + S \sin \psi)\mathbf{j} , \\ \mathbf{T}_b = T_2\mathbf{b} + S\mathbf{a} = (T_2 \cos \psi + S \cos \varphi)\mathbf{i} + (T_2 \sin \psi + S \sin \varphi)\mathbf{j} , \end{cases} \tag{2.7}$$

where T_1 and T_2 are tensions in the X_1- and X_2-cords and S is shear stress. In order to satisfy the equilibrium of moment on an arc (see Pipkin [3]), the shear components of $\mathbf{T}_a$ and $\mathbf{T}_b$ have been chosen to be the same. The sign of shear stress S is assumed to be consistent with the convention of classical continuum mechanics.

Constitutive relations are next stated as follows. The tensions T_1 and T_2 are related to the stretches λ_1 and λ_2 by

$$T_\alpha = \begin{cases} T_\alpha(\lambda_\alpha) > 0 & \text{if } \lambda_\alpha > 1 \\ 0 & \text{if } \lambda_\alpha \leqslant 1 \end{cases} \qquad (\alpha = 1, 2) , \tag{2.8}$$

or

$$\lambda_\alpha \begin{cases} = \lambda_\alpha(T_\alpha) > 1 & \text{if } T_\alpha > 0 \\ \leqslant 1 & \text{if } T_\alpha = 0 \end{cases} \qquad (\alpha = 1, 2) . \tag{2.9}$$

Also the shear stress S is related uniquely to shear angle $\pi/2-(\psi-\varphi)$ by

$$S=S^*(\pi/2-\psi+\varphi)=S(\psi-\varphi)\begin{cases} >0 & \text{if } 0\leq\psi-\varphi<\pi/2\,,\\ =0 & \text{if } \psi-\varphi=\pi/2\,,\\ <0 & \text{if } \pi\geq\psi-\varphi>\pi/2\,. \end{cases} \tag{2.10}$$

The conditions in (2.8) and (2.9) follow from the assumption that the cords cannot withstand compression. When $\lambda_1\leq 1$ and/or $\lambda_2\leq 1$, the X_1-cords and/or X_2-cords are not stretched and $T_1=0$ and/or $T_2=0$. We refer to the region in which the cords are not stretched as slack regions. It can be seen from (2.9) that the stretches in a slack region cannot be determined by the corresponding tension and from (2.10) that $S(\psi-\varphi)$ is a decreasing function of $\psi-\varphi$ in the range $0\leq\psi-\varphi\leq\pi$.

It can be seen that the Piola (or nominal) stress tensor is given by

$$\begin{pmatrix} T_1\cos\varphi+S\cos\psi & T_1\sin\varphi+S\sin\psi\\ T_2\cos\psi+S\cos\varphi & T_2\sin\psi+S\sin\varphi \end{pmatrix}.$$

Then the equilibrium equations are of the form

$$\begin{aligned} &\frac{\partial}{\partial X_1}(T_1\cos\varphi+S\cos\psi)+\frac{\partial}{\partial X_2}(T_2\cos\psi+S\cos\varphi)=0\,,\\ &\frac{\partial}{\partial X_1}(T_1\sin\varphi+S\sin\psi)+\frac{\partial}{\partial X_2}(T_2\sin\psi+S\sin\varphi)=0\,. \end{aligned} \tag{2.11}$$

Here we have assumed that no body force exists. These equations can also be derived by considering the equilibrium of an infinitesimal rectangular element of the membrane (see Shi [7]).

Here we have seven unknowns T_1, λ_1, φ, T_2, λ_2, ψ and S with four equations (2.6) and (2.11) and three relations (2.8) (or (2.9)) and (2.10). So the problem is completed by adding boundary conditions. Substituting (2.8) and (2.10) into (2.11), we have four equations for four unknowns λ_1, λ_2, φ and ψ. Furthermore, expressing λ_1, λ_2, φ and ψ in terms of x_1 and x_2, with aid of (2.4) and (2.5), and substituting the results into the equilibrium equations (2.11) gives two governing equations for x_1 and x_2. These equations should be solved when the conditions, specially the displacement, along the boundary is specified. The compatibility equations (2.6) are satisfied provided that x_1 and x_2 are continuous.

For a traction boundary value problem, we may employ an alternative approach. We can obtain four equations for four unknowns T_1, T_2, φ and ψ by substituting (2.9) into (2.6) and (2.10) into (2.11). We might be able to reduce the number of the equations further by introducing the stress functions $F_1(X_1,X_2)$ and $F_2(X_1,X_2)$ such that

$$\begin{cases} T_1\cos\varphi+S\cos\psi=F_{1,2}\,, & T_2\cos\psi+S\cos\varphi=-F_{1,1}\,,\\ T_1\sin\varphi+S\sin\psi=F_{2,2}\,, & T_2\sin\psi+S\sin\varphi=-F_{2,1}\,. \end{cases} \tag{2.12}$$

Thus the equilibrium equations (2.11) are satisfied identically. From the equations in (2.12), we express T_1, T_2, and S in terms of F_1, F_2, φ and ψ as

$$T_1=\frac{F_{1,2}\sin\psi-F_{2,2}\cos\psi}{\sin(\psi-\varphi)}\,, \tag{2.13a}$$

$$T_2 = \frac{F_{1,1} \sin \varphi - F_{2,1} \cos \varphi}{\sin(\psi - \varphi)} , \tag{2.13b}$$

$$S = -\frac{F_{1,1} \sin \psi - F_{2,1} \cos \psi}{\sin(\psi - \varphi)} , \tag{2.14a}$$

or

$$S = -\frac{F_{1,2} \sin \varphi - F_{2,2} \cos \varphi}{\sin(\psi - \varphi)} , \tag{2.14b}$$

if $\sin(\psi - \varphi) \neq 0$. The special situation in which $\sin(\psi - \varphi) = 0$ will be discussed in Section 3. We know the function $S(\psi - \varphi)$. If we can solve φ and ψ from (2.14) in terms of F_1 and F_2, then substituting into (2.13) gives T_1 and T_2, hence λ_1 and λ_2 through (2.9), in terms of F_1 and F_2 only. For example, if

$$S = G \cos(\psi - \varphi)/\sin(\psi - \varphi) \tag{2.15}$$

with G being constant, then a solution for (2.14) is given by

$$\begin{aligned} &\sin \varphi = -F_{1,1}/G , \qquad \cos \varphi = F_{2,1}/G , \\ &\sin \psi = -F_{1,2}/G , \qquad \cos \psi = F_{2,2}/G . \end{aligned} \tag{2.16}$$

Finally introducing the resulting expressions into the compatibility conditions (2.6), we obtain the governing equations for F_1 and F_2. After solving these equations, (2.4) and (2.5) may then be integrated to yield x_1 and x_2.

For this problem, it is necessary to express the stretches λ_1 and λ_2 as functions of T_1 and T_2 respectively. It follows from the requirements in (2.8) and (2.9) that in slack region in which $T_1 = 0$ it is not possible to express λ_1 as a function of T_1 and that in slack region in which $T_2 = 0$ the stretch λ_2 is not expressible as a function of T_2. Thus it is by no means evident that a solution will exist to any specified traction boundary value problem and even if a solution exists it may well not be unique. So we may encounter difficulties when attempting a numerical analysis of this problem. The question of existence and uniqueness of solutions is not our main interest in this paper but rather we seek some special solutions in which the entire region is either fully-stretched (Sections 3, 4, 5) or slack (Section 6).

The simplest fully-stretched deformation is the homogeneous one in which all the quantities T_1, λ_1, φ, T_2, λ_2, ψ and S are constant throughout the membrane and equal to the uniform values along the boundary.

We have assumed that $S(\pi/2) = 0$, so the solutions obtained for the membranes without shear resistance (e.g., see Green and Shi [8, 9]) in which $\psi - \varphi = \pi/2$ everywhere are the solutions for the present membrane. For example, when the relations (2.8) are of the linear form $T_\alpha = E_\alpha(\lambda_\alpha - 1)$, $(\alpha = 1, 2)$, and one family of the cords is deformed into the radial directions and the other into the circumferential directions of a circle, the equations outlined in the present section are satisfied.

3. Degenerate deformations and deformations with constant tensions

In this section we assume that the constitutive relation (2.10) gives $-\infty < S(\pi) < S(0) < \infty$, i.e. the cases $\psi = \varphi$ and $\varphi = \pi + \varphi$ are allowed. First we discuss a deformation in which

$\psi = \varphi$ or $\psi = \pi + \varphi$ in some region. If there exists such a deformation, then the two cords through any point (X_1, X_2) in the region before the deformation must be parallel to each other after deformation and the region collapses into a single curve. Following Green and Shi [10] and choosing first the case $\psi = \varphi$ then the equilibrium equations (2.11) become

$$\begin{aligned} &\frac{\partial}{\partial X_1}[(T_1 + S_0)\cos\varphi] + \frac{\partial}{\partial X_2}[(T_2 + S_0)\cos\varphi] = 0\,, \\ &\frac{\partial}{\partial X_1}[(T_1 + S_0)\sin\varphi] + \frac{\partial}{\partial X_2}[(T_2 + S_0)\sin\varphi] = 0\,, \end{aligned} \tag{3.1}$$

whilst the compatibility conditions (2.6) reduce to

$$\begin{aligned} &\frac{\partial}{\partial X_2}(\lambda_1\cos\varphi) - \frac{\partial}{\partial X_1}(\lambda_2\cos\varphi) = 0\,, \\ &\frac{\partial}{\partial X_2}(\lambda_1\sin\varphi) - \frac{\partial}{\partial X_1}(\lambda_2\sin\varphi) = 0\,. \end{aligned} \tag{3.2}$$

Here the shear stress $S_0 = S(0)$ is constant. Expanding (3.1) and (3.2), and then rearranging them, we have

$$\begin{aligned} &\left(\frac{\partial T_1}{\partial X_1} + \frac{\partial T_2}{\partial X_2}\right)\cos\varphi - \left[(T_1 + S_0)\frac{\partial\varphi}{\partial X_1} + (T_2 + S_0)\frac{\partial\varphi}{\partial X_2}\right]\sin\varphi = 0\,, \\ &\left(\frac{\partial T_1}{\partial X_1} + \frac{\partial T_2}{\partial X_2}\right)\sin\varphi + \left[(T_1 + S_0)\frac{\partial\varphi}{\partial X_1} + (T_2 + S_0)\frac{\partial\varphi}{\partial X_2}\right]\cos\varphi = 0\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\left(\frac{\partial \lambda_1}{\partial X_2} - \frac{\partial \lambda_2}{\partial X_1}\right)\cos\varphi - \left(\lambda_1\frac{\partial\varphi}{\partial X_2} - \lambda_2\frac{\partial\varphi}{\partial X_1}\right)\sin\varphi = 0\,, \\ &\left(\frac{\partial \lambda_1}{\partial X_2} - \frac{\partial \lambda_2}{\partial X_1}\right)\sin\varphi + \left(\lambda_1\frac{\partial\varphi}{\partial X_2} - \lambda_2\frac{\partial\varphi}{\partial X_1}\right)\cos\varphi = 0\,, \end{aligned} \tag{3.4}$$

and these may be rewritten as

$$\frac{\partial T_1}{\partial X_1} + \frac{\partial T_2}{\partial X_2} = 0\,, \qquad (T_1 + S_0)\frac{\partial\varphi}{\partial X_1} + (T_2 + S_0)\frac{\partial\varphi}{\partial X_2} = 0\,, \tag{3.5}$$

$$\frac{\partial \lambda_1}{\partial X_2} - \frac{\partial \lambda_2}{\partial X_1} = 0\,, \qquad \lambda_1\frac{\partial\varphi}{\partial X_2} - \lambda_2\frac{\partial\varphi}{\partial X_1} = 0\,. \tag{3.6}$$

If the collapsed region carries non-zero load then $\lambda_1(T_1 + S_0) + \lambda_2(T_2 + S_0) \neq 0$ and (3.5b) and (3.6b) have solution

$$\frac{\partial\varphi}{\partial X_1} = \frac{\partial\varphi}{\partial X_2} = 0$$

or

$$\varphi \equiv \text{constant} = \varphi_0\,. \tag{3.7}$$

Equation (3.7) shows that the region collapses into a straight line. It follows that the tractions applied to the boundary of the collapse region must everywhere be parallel to this line.

In a similar procedure, we find that if in some region $\psi = \pi + \varphi$, then the equations corresponding to (3.5a) and (3.6a) are

$$\frac{\partial T_1}{\partial X_1} - \frac{\partial T_2}{\partial X_2} = 0, \qquad \frac{\partial \lambda_1}{\partial X_2} + \frac{\partial \lambda_2}{\partial X_1} = 0, \tag{3.8}$$

and that region also has to collapse into a straight line.

The tensions and stretches in the collapsed region can be determined from (3.5a) and (3.6a), or from (3.8), with the aid of the response functions

$$T_1 = T_1(\lambda_1), \qquad T_2 = T_2(\lambda_2), \tag{3.9}$$

or their inverses

$$\lambda_1 = \lambda_1(T_1), \qquad \lambda_2 = \lambda_2(T_2). \tag{3.10}$$

These equations do not involve the shear stress and are the same as those in [10] so the solutions found there can also be applied here. More details are referred to [10].

Now we turn to the second part of this section; assume that the tensions T_α (or the stretches λ_α) ($\alpha = 1, 2$) are constant throughout the region occupied by the membrane and seek for the deformation which satisfies the governing equations. In this case the compatibility conditions (2.6) reduce to

$$\begin{aligned} \lambda_2 \sin \psi \frac{\partial \psi}{\partial X_1} &= \lambda_1 \sin \varphi \frac{\partial \varphi}{\partial X_2}, \\ \lambda_2 \cos \psi \frac{\partial \psi}{\partial X_1} &= \lambda_1 \cos \varphi \frac{\partial \varphi}{\partial X_2}. \end{aligned} \tag{3.11}$$

It follows from (3.11) that

$$\sin(\psi - \varphi) = 0, \qquad \lambda_2 \frac{\partial \psi}{\partial X_1} = \pm \lambda_1 \frac{\partial \varphi}{\partial X_2} \tag{3.12}$$

or $\sin(\varphi - \varphi) \neq 0$ with

$$\frac{\partial \psi}{\partial X_1} = 0, \qquad \frac{\partial \varphi}{\partial X_2} = 0. \tag{3.13}$$

The first of (3.12) implies that the membrane collapses into a single curve and the preceding discussion shows that the angles φ and ψ are constant throughout the region. Therefore the collapsed deformation is homogeneous.

The equations (3.13) imply that

$$\varphi = \varphi(X_1), \qquad \psi = \psi(X_2).$$

The equilibrium equations (2.11) then become

$$(T_1 \sin\varphi + S' \cos\psi)\frac{d\varphi}{dX_1} + (T_2 \sin\psi - S' \cos\varphi)\frac{d\psi}{dX_2} = 0\,,$$
$$(T_1 \cos\varphi - S' \sin\psi)\frac{d\varphi}{dX_1} + (T_2 \cos\psi + S' \sin\varphi)\frac{d\psi}{dX_2} = 0\,, \tag{3.14}$$

where the prime denotes the differentiation with respect to the argument $\chi = \psi - \varphi$. The determinant of the coefficients of $d\varphi/dX_1$ and $d\psi/dX_2$ in (3.14) is

$$(T_1 \sin\varphi + S' \cos\psi)(T_2 \cos\psi + S' \sin\varphi) - (T_1 \cos\varphi - S' \sin\psi)(T_2 \sin\psi - S' \cos\varphi)$$
$$= (T_1 + T_2)S' - (T_1T_2 + S'^2)\sin(\psi - \varphi)\,. \tag{3.15}$$

Since the function $S(\chi)$ decreases (i.e. $S' < 0$) and $\sin\chi > 0$ in the range $0 < \chi < \pi$, the above expression is strictly negative. Therefore it follows from (3.14) that

$$\frac{d\varphi}{dX_1} = 0\,, \qquad \frac{d\psi}{dX_2} = 0\,,$$

which imply

$$\varphi \equiv \text{constant}\,, \qquad \psi \equiv \text{constant}\,,$$

throughout the region. With constant tensions and stretches, the deformation is homogeneous too. Therefore we can conclude that if the tensions (or the stretches) are constant throughout the membrane then the deformation must be homogeneous.

4. Half-universal state of tensions

The deformation we consider here is that in which $\psi = \pi/2$ and $\varphi = \varphi(X_1)$, i.e. the X_2-cords remain straight and parallel to the x_2-axis and the X_1-cords lie along a family of congruent curves. For this deformation, φ is the shear angle so $S = S^*(\varphi) = S(\pi/2 - \varphi) = \bar{S}(X_1)$ and the equations (2.6) and (2.11) become

$$\cos\varphi\,\frac{\partial\lambda_1}{\partial X_2} = 0\,, \tag{4.1}$$

$$\sin\varphi\,\frac{\partial\lambda_1}{\partial X_2} - \frac{\partial\lambda_2}{\partial X_1} = 0\,, \tag{4.2}$$

$$\frac{\partial}{\partial X_1}(T_1 \cos\varphi) = 0\,, \tag{4.3}$$

$$\frac{\partial}{\partial X_1}(T_1 \sin\varphi + S) + \frac{\partial T_2}{\partial X_2} = 0\,. \tag{4.4}$$

For the membrane which does not collapse, i.e. $\varphi \neq \psi = \pi/2$, or $\varphi \neq \psi - \pi = -\pi/2$, (4.1) and (4.2) give

$$\lambda_1 = \lambda_1(X_1)\,, \qquad \lambda_2 = \lambda_2(X_2)\,. \tag{4.5}$$

With the aid of (2.8), (4.3) and (4.4) produce

$$T_1 \cos\varphi = T_{1c}\,, \tag{4.6}$$

$$T_1 \sin\varphi + S = -KX_1 + A\,, \tag{4.7}$$

$$T_2 = KX_2 + B\,, \tag{4.8}$$

where T_{1c}, K, A and B are constant.

If we give an appropriate relation $S = S(\pi/2 - \varphi)$, we can solve for T_1 and φ from (4.6) and (4.7) in terms of X_1. Then in turn the shear stress S can be expressed in terms of X_1. For instance, we choose Pipkin's [3] relation, for inextensible networks, between deformation and shear stress

$$S = G\tan\varphi\,, \tag{4.9}$$

where the constant G is shear modulus. Then from (4.6) and (4.7), we have

$$\tan\varphi = \frac{-KX_1 + A}{T_{1c} + G}\,, \tag{4.10}$$

$$T_1 = \frac{T_{1c}}{T_{1c} + G}\,[(T_{1c} + G)^2 + (-KX_1 + A)^2]^{1/2}\,. \tag{4.11}$$

The constants A, B, T_{1c} and K should be determined from boundary conditions. Since they are associated directly with the state of tensions it is easy to determine them when suitable tractions are specified along the boundary. For example, if the membrane occupies the region $0 \leq X_1 \leq L$, $0 \leq X_2 \leq H$ and we specify the tractions such that

$$\left\{\begin{array}{l}\varphi = 0\\ T_1 = T_1^*\end{array}\right. \quad \text{at } X_1 = 0\,,$$

$$T_2 = 0 \qquad \text{at } X_2 = 0\,,$$

$$T_2 = T_2^* \qquad \text{at } X_2 = H\,,$$

then

$$\begin{aligned} &T_2 = T_2^* X_2/H\,,\\ &T_1 = T_1^*\left[1 + \frac{T_2^{*2}X_1^2}{(T_1^* + G)^2H^2}\right]^{1/2},\\ &\tan\varphi = -T_2^* X_1/[(T_1^* + G)H]\,,\\ &S = -GT_2^* X_1/[(T_1^* + G)H]\,. \end{aligned} \tag{4.12}$$

We have completely determined the tensions T_1 and T_2 and their directions as well as the shear stress without specifying the forms of stretch-tension relations. But we used (4.9), so we call this state a half-universal state of tensions. It can be seen that both $T_1 > 0$ and $T_2 > 0$ inside the region. So we can calculate λ_1 and λ_2 when the relations in (2.9) are given, and then integrate (2.4) and (2.5) to find x_1 and x_2.

5. Straight line deformation

In this section we consider membranes with stretch-tension relations

$$T_\alpha = \begin{cases} E_\alpha(\lambda_\alpha - 1) & \text{if } \lambda_\alpha > 1 \\ 0 & \text{if } \lambda_\alpha \leq 1 \end{cases} \quad (\alpha = 1, 2)\,, \tag{5.1}$$

and shear-deformation relation

$$S = G\,\frac{\cos(\psi - \varphi)}{\sin^2(\psi - \varphi)}\,, \tag{5.2}$$

where E_1, E_2 and G are positive constants. These relations are chosen so that we can obtain an analytic solution for the governing equations. The relation (5.2) states that infinite shear stress is needed to collapse a finite region of membrane into a segment of curve, i.e. $\psi = \varphi$.

In the deformation considered here, we assume that $\psi = \pi/2$ and $\varphi = \varphi(X_2)$, i.e. the X_2-cords remain straight and parallel to x_2-axis and the X_1cords just remain straight. Then the relation (5.2) becomes

$$S = G\,\frac{\sin\varphi(X_2)}{\cos^2\varphi(X_2)} \tag{5.3}$$

and the equations (2.6) and (2.11) become

$$\frac{\partial}{\partial X_2}(\lambda_1 \cos\varphi) = 0\,, \tag{5.4}$$

$$\frac{\partial}{\partial X_2}(\lambda_1 \sin\varphi) - \frac{\partial \lambda_2}{\partial X_1} = 0\,, \tag{5.5}$$

$$\cos\varphi\,\frac{\partial T_1}{\partial X_1} + \frac{\partial}{\partial X_2}(S\cos\varphi) = 0\,, \tag{5.6}$$

$$\sin\varphi\,\frac{\partial T_1}{\partial X_1} + \frac{\partial}{\partial X_2}(T_2 + S\sin\varphi) = 0\,. \tag{5.7}$$

It follows, from (5.4), that

$$\lambda_1 = f(X_1)/\cos\varphi(X_2)\,, \tag{5.8}$$

where $f(X_1)$ is an arbitrary function of the argument X_1. The equation (5.8) is valid provided $\varphi \neq \pi/2$, which is the requirement that the membrane does not collapse. Then (5.5) gives

$$\lambda_2 = F(X_1)\,\frac{d(\tan\varphi)}{dX_2} + g(X_2)\,, \tag{5.9}$$

where $F(X_1) = \int f(X_1)\,dX_1$ and $g(X_2)$ is another arbitrary function of its argument.

Substituting (5.8) and (5.9) into (5.1), then together with (5.3) into (5.6) and (5.7), we have

$$E_1 f'(X_1) + G\,\frac{d(\tan\varphi(X_2))}{dX_2} = 0\,, \tag{5.10}$$

$$E_1 f'(X_1)\tan\varphi + E_2 F(X_1)\,\frac{d^2(\tan\varphi)}{dX_2^2} + G\,\frac{d(\tan^2\varphi)}{dX_2} + E_2 g'(X_2) = 0\,. \tag{5.11}$$

The equation (5.10) requires that

$$f(X_1) = \frac{-K}{E_1}\,X_1 + A\,, \tag{5.12}$$

$$\tan\varphi(X_2) = \frac{K}{G}\,X_2 + B\,, \tag{5.13}$$

where A, B and K are constant. Then integration of (5.11) gives

$$g(X_2) = -\,\frac{K^2}{2E_2G}\,X_2^2 - \frac{K}{E_2}\,BX_2 + C\,. \tag{5.14}$$

Here C is another constant.

With the aid of $\cos^2\varphi(1+\tan^2\varphi) = 1$, substitution (5.12), (5.13) and (5.14) into (5.8) and (5.9) gives

$$\lambda_1 = (A - KX_1/E_1)[1 + (B + KX_2/G)^2]^{1/2}\,, \tag{5.15}$$

$$\lambda_2 = -\,\frac{K^2}{2G}\left(\frac{X_1^2}{E_1} + \frac{X_2^2}{E_2}\right) + K\left(\frac{A}{G}\,X_1 - \frac{B}{E_2}\,X_2\right) + D\,, \tag{5.16}$$

where D is a constant.

The constants A, B, C, D and K should be determined from the conditions along the boundary. They are related to the angle φ and the stretch ratios. The latter are in turn related to the tensions algebraically but differentially to the deformation. So it is easier to determine them from traction boundary conditions than displacement conditions. If the membrane occupies the region $0 \leqslant X_1 \leqslant L$ and $0 \leqslant X_2 \leqslant H$ and we specify

$$\varphi = 0 \qquad \text{at } X_2 = 0\,,$$

$$\varphi = \varphi^* \qquad \text{at } X_2 = H\,,$$

$$\begin{cases} \lambda_1 = \lambda_1^* \\ \lambda_2 = \lambda_2^* \end{cases} \quad \text{at } X_1 = X_2 = 0\,,$$

then

$$\begin{aligned}
\tan\varphi &= \frac{X_2}{H}\tan\varphi^*\,,\\
\lambda_1 &= \left(\lambda_1^* - \frac{GX_1}{E_1H}\tan\varphi^*\right)\left(1+\frac{X_2^2}{H^2}\tan^2\varphi^*\right)^{1/2},\\
\lambda_2 &= \lambda_2^* + \frac{X_1}{H}\lambda_1^*\tan\varphi^* - \left(\frac{X_1^2}{E_1}+\frac{X_2^2}{E_2}\right)\frac{G}{2H^2}\tan^2\varphi^*\,,\\
S &= \frac{X_2}{H}\left(1+\frac{X_2^2}{H^2}\tan^2\varphi^*\right)^{1/2} G\tan\varphi^*\,.
\end{aligned} \tag{5.17}$$

It can be seen that if $\varphi^*=0$ then $\varphi=0$, $\lambda_1=\lambda_1^*$, $\lambda_2=\lambda_2^*$ and $S=0$ everywhere. Furthermore for $0<\varphi^*<\pi/2$, λ_1 decreases as X_1 increases and increases as X_2 does. So to ensure that $\lambda_1>1$ inside the region, we must have $\lambda_1^*\geq 1+(GL\tan\varphi^*)/(E_1H)$. Thus λ_2 increases as X_1 and decreases as X_2 increases. The condition for $\lambda_2\geq 1$ inside the region is $\lambda_2^*\geq 1+(G\tan^2\varphi^*)/(2E_2)$. Then the deformation functions x_1 and x_2 can be found by integrating (2.4) and (2.5) and the tensions from (2.8). To maintain the deformation, the traction along the boundary must be determined from the resulting expressions for the tensions. The decreases of the stretches and tensions as X_1 or X_2 increases are caused by the presence of the shear stress S.

6. Deformations in slack regions

Some special deformations in slack regions will be considered in this section. We first discuss the deformations in half-slack region in which $T_2>0$ but $T_1=0$, i.e. $\lambda_1\leq 1$. We assume that the stretched X_2-cords are straight and parallel to each other, i.e. $\psi=\psi_0=$ constant. We also assume that the region does not collapse, that is, $\varphi\neq\psi_0$ or $\psi_0+\pi$. Then the equilibrium equations in (2.11) become

$$\begin{aligned}
\left(\frac{\partial S}{\partial X_1}+\frac{\partial T_2}{\partial X_2}\right)\cos\psi_0 + \frac{\partial}{\partial X_2}(S\cos\varphi) &= 0\,,\\
\left(\frac{\partial S}{\partial X_1}+\frac{\partial T_2}{\partial X_2}\right)\sin\psi_0 + \frac{\partial}{\partial X_2}(S\sin\varphi) &= 0\,,
\end{aligned} \tag{6.1}$$

and the compatibility conditions (2.6) are

$$\begin{aligned}
\frac{\partial}{\partial X_2}(\lambda_1\cos\varphi) &= \frac{\partial\lambda_2}{\partial X_1}\cos\psi_0\,,\\
\frac{\partial}{\partial X_2}(\lambda_1\sin\varphi) &= \frac{\partial\lambda_2}{\partial X_1}\sin\psi_0\,.
\end{aligned} \tag{6.2}$$

We have the relation

$$T_2=T_2(\lambda_2)\quad\text{or}\quad \lambda_2=\lambda_2(T_2)\,, \tag{6.3}$$

for the X_2-cords, but not for the X_1-cords. The shear stress S depends upon φ alone, say, given by

$$S = S(\varphi) . \tag{6.4}$$

It follows from (6.1) that

$$\frac{\partial}{\partial X_2} [S(\varphi) \sin(\psi_0 - \varphi)] = 0 .$$

Therefore

$$\frac{\partial \varphi}{\partial X_2} = 0 ,$$

that is, $\varphi = \varphi(X_1)$, a function of X_1 alone. This implies that the X_1-cords lie along a family of congruent curves and this is a special case of those discussed in Section 4. Then, with the relation (6.4), the equilibrium equations become

$$\frac{\mathrm{d}S}{\mathrm{d}X_1} + \frac{\partial T_2}{\partial X_2} = 0 . \tag{6.5}$$

Furthermore, the compatibility conditions (6.2) yield

$$\frac{\partial \lambda_1}{\partial X_2} = 0 , \qquad \frac{\partial \lambda_2}{\partial X_1} = 0$$

since $\sin(\psi_0 - \varphi) \neq 0$. These are equivalent to

$$\lambda_1 = \lambda_1(X_1) , \qquad \lambda_2 = \lambda_2(X_2) . \tag{6.6}$$

With the relation (6.3), the equation (6.5) becomes

$$\frac{\mathrm{d}S}{\mathrm{d}X_1} = - \frac{\mathrm{d}T_2}{\mathrm{d}X_2} .$$

Since the left-hand side depends on X_1 alone while the right-hand side on X_2 alone, we can derive

$$\frac{\mathrm{d}S}{\mathrm{d}X_1} = K , \qquad \frac{\mathrm{d}T}{\mathrm{d}X_2} = -K ,$$

with K being constant. So we integrate them and find

$$\begin{aligned} S &= KX_1 + A , \\ T_2 &= -KX_2 + B , \end{aligned} \tag{6.7}$$

where A and B are constant, which can be determined from the conditions specified along the boundary.

When the relation (6.4) is specified and if we can solve φ from it in terms of S, then we can find $\varphi = \varphi(X_1)$. Thus we find the state of stress. The stretch ratio λ_2 is given by (6.4) with the second of (6.7). Due to no more equation to be satisfied, $\varphi(X_1)$ and $\lambda_1(X_1)$ are arbitrary, but $\lambda_1(X_1)$ must be less than unity. They can be determined if their values are given along a X_2-cord.

A special case occurs when the X_1-cords remain straight and parallel to each other too. In this case $\varphi = \varphi_0$, a constant different from ψ_0. Then the shear stress S is constant too, which implies that the constant K in (6.7) must be zero. Therefore the tension T_2 is constant in the region.

Now we assume that in the half-slack region $\varphi = \psi_0$; the region collapses into a single straight line. The shear stress S becomes constant in the region and the equilibrium equations produce

$$\frac{\partial T_2}{\partial X_2} = 0 ,$$

that is

$$T_2 = T_2(X_1) , \tag{6.8}$$

which, with the aid of (6.3), yields

$$\lambda_2 = \lambda_2(X_1) . \tag{6.9}$$

The compatibility conditions become

$$\frac{\partial \lambda_1}{\partial X_2} = \frac{\mathrm{d}\lambda_2}{\mathrm{d}X_1} . \tag{6.10}$$

Integrating with respect to X_2, we have

$$\lambda_1 = X_2 \lambda_2'(X_1) + f(X_1) , \tag{6.11}$$

where $f(X_1)$ is an arbitrary function of its argument. No more equation inside the region is to be satisfied, $\lambda_2(X_1)$ is arbitrary too, which, together with $f(X_1)$, should be determined from the displacement conditions along the boundary, provided they produce $\lambda_1 < 1$.

Finally, we discuss the deformations in a fully-slack region in which $T_1 = 0$ and $T_2 = 0$ with assumption that $\psi = \psi_0$, a constant. As before, the shear stress S depends upon φ, given by $S(\varphi)$. The equilibrium equations are reduced to

$$\begin{aligned} &\frac{\partial S}{\partial X_1} \cos \psi_0 + \frac{\partial}{\partial X_2} (S \cos \varphi) = 0 , \\ &\frac{\partial S}{\partial X_1} \sin \psi_0 + \frac{\partial}{\partial X_2} (S \sin \varphi) = 0 . \end{aligned} \tag{6.12}$$

If $\sin(\psi_0 - \varphi) \neq 0$, the equations in (6.12) produce

$$\frac{\partial \varphi}{\partial X_2} = 0 , \qquad \frac{\partial S}{\partial X_1} = 0 .$$

Since $S = S(\varphi)$, we find that φ must be a constant in the region. Then the compatibility conditions yield

$$\lambda_1 = \lambda_1(X_1)\,, \qquad \lambda_2 = \lambda_2(X_2)\,. \tag{6.13}$$

Here $\lambda_1(X_1)$ and $\lambda_2(X_2)$ are arbitrary functions which should be determined from the displacement conditions along boundary and less than unity.

If $\sin(\psi_0 - \varphi) = 0$, then directly φ is a constant, the equilibrium equations (6.12) are satisfied and the compatibility conditions produce again (6.13). Thus in a fully-slack region if one family of cords remain straight and parallel, then the other family must be straight and parallel too.

We note that the solutions found in this section are valid for general relations between the stress and deformation.

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